ON THE ENUMERATION OF PROPER AND IMPROPER REPRESENTATIONS IN HOMOGENEOUS FORMS.

By E. T. Bell.

1. By the usual definition, a particular representation of n in the form

$$\Sigma a_{ij}x_ix_j, \ (i, j = 1, 2, \cdots, r), \tag{1}$$

e.g., through $(x_1, x_2, \cdots, x_r) = (x_1', x_2', \cdots, x_r')$, is proper or improper according as the G.C.D. of the x' is $\equiv 1$; and two representations $(x_1', x_2', \cdots, x_r')$, $(x_1'', x_2'', \cdots, x_r'')$ are identical only when

$$x_i' = x_i'', \ (i = 1, \cdots, r).$$

Let $T(n)$, $P(n)$ denote respectively the total number of representations, and the number of proper representations of n in (1). Then, the Σ referring to every positive d such that n/d^2 is an integer, we have* directly from the definitions

$$T(n) = \Sigma P(n/d^2). \tag{2}$$

Similarly, if in place of (1) we have a homogeneous form of degree s, there is between the corresponding T, P the relation

$$T(n) = \Sigma P(n/d^s), \tag{3}$$

the Σ extending to all positive d such that n/d^s is an integer. Practically nothing of importance, except in the case of binary cubics, being known concerning T, P when $s > 2$, we shall confine the discussion to (2), merely indicating in § 15 the nature of the easy extension whereby all of the general formulæ for $s = 2$ can be carried up to $s > 2$ whenever specific theorems for the latter cases shall be available.

2. If the strictly arithmetical theory, due principally to Eisenstein, H. J. S. Smith and Minkowski, be used to find $T(n)$, $P(n)$, the natural (and historical) order appears to be first the determination of $P(n)$, and thence by (2) the deduction of $T(n)$. If, however we seek $T(n)$, $P(n)$ algebraically, either by elliptic functions or otherwise (cf. § 9), $T(n)$ always appears first, $P(n)$ entering, if at all, only through cumbersome and artificial transformations of the analysis appropriate to $T(n)$. It

* For a detailed discussion of a particular case, cf. Bachmann, *Die Arithmetik der Quadratischen Formen*, p. 602. Note that in accordance with modern usage we have not assumed $a_{ij} = a_{ji}$ in (1).

seems in fact that without previous knowledge of the results to be attained, suitable transformations of the fundamental identities would present themselves but seldom. On the other hand, it has frequently been pointed out* that where solutions of arithmetical problems by elliptic functions or other algebraic means exist, they are in general much simpler, shorter, and less delicate in application than the corresponding investigations by processes peculiar to the theory of numbers. Hence it is of some importance to invert the present problem, deducing the $P(n)$ directly from the $T(n)$, the latter in many instances being given very simply by algebraical methods. It will be seen in §§ 6, 7 how this may always be done, and how, in §§ 8–11, for one of the most important classes of representations the general formulæ assume simple and interesting shapes. By means of the formulæ developed the deduction of $P(n)$ from $T(n)$ is immediate; in §§ 12–14 a few illustrations are given in the derivation of $P(n)$ theorems due to Eisenstein and Liouville, the proofs for which seem not to have been published hitherto; also in writing down some new results for 6, 8, 10 and 12 squares. But the object of this paper being the general method, and not special consequences, applications are included only in sufficient number to make clear the use of the formulæ. It may be mentioned that by means of the formulæ in §§ 10, 11 all of Liouville's numerous $P(n)$ theorems which he published without proofs in the second series of his Journal, may be demonstrated almost at a glance. His $T(n)$ theorems were all proved by Pepin, cf. § 9; the proofs of the rest, all of which have been found by the methods of this paper, will appear in the Journal de Mathematiques for 1919. Theorems such as those of Liouville and Eisenstein concerning special quadratic forms are of importance as guides in the general theory of (1), which still is far from complete.

3. All letters m, n, d, δ denote positive non-zero integers; *the m's are always odd, and the n's arbitrary.* We define n to be simple if it is divisible by no square > 1; and adopting Sylvester's convenient term, call the number of distinct prime factors of n its multiplicity. Consider

$$(4) \qquad F(1) = 1, \qquad F(n_1n_2) = F(n_1)F(n_2), \qquad D(n_1, n_2) = 1,$$

where $D(n_1, n_2)$ is the G.C.D. of n_1, n_2. Functions $F, f, g, \cdots$ satisfying (4) we shall call factorable. If by the nature of the function, factorable $f(n)$ is undefined for $n = 1$, then by convention $f(1) = 1$; also, $f(x) = 0$ when x is not a positive integer. We shall require Möbius' $\mu(n)$, which $= 0$ if n is not simple, and which otherwise $= +1$ or -1 according as the multiplicity of n is even or odd. It is readily seen that $\mu(n)$ is factor-

* See, for example, Glaisher's remarks in the Proc. London Math. Society (2), 5 (1907), pp. 489–490, § 16.

able, and that

$$(5) \qquad \sum_n \mu(d) = 0, \; n > 1,$$

the notation $\sum_n$ indicating that the sum is taken with respect to all divisors d of n. The fundamental property (5) is well known; nevertheless we recall one of the shortest ways by which it may be established, as the same applies to all subsequent identities concerning factorable functions. Since $\mu(n)$ is factorable, it suffices to verify (5) for $n = p^a$, where p is prime. The divisors d in this case are $1, p, p^2, \cdots, p^a$; and from the definition of $\mu(n)$: $\mu(1) = 1$; $\mu(p) = -1$; $\mu(p^a) = 0$, $a > 1$. In the same way each of the factorable function identities given later may be proved by verifying them for $n = p^a$ directly from the definitions of the particular functions involved. These verifications, presenting no difficulty or interest, will be omitted.

4. Let (d, δ) denote any pair of conjugate divisors of n, so that $n = d\delta$. Form the value of $\varphi_1(x)\psi(y)$ for $(x, y) = (d, \delta)$, sum $\varphi_1(d)\psi(\delta)$ over all pairs (d, δ), and denote the result by $\sum_n \varphi_1(d)\psi(\delta)$. Let (δ_1, δ_2) denote any pair of conjugate divisors of δ, so that

$$n = d\delta, \qquad \delta = \delta_1\delta_2, \qquad n = d\delta_1\delta_2.$$

Put $\psi(n) = \sum_n \varphi_2(d)\varphi_3(\delta)$; whence,

$$\sum_n \varphi_1(d)\psi(\delta) = \sum_n [\varphi_1(d) \sum_\delta \varphi_2(\delta_1)\varphi_3(\delta_2)] = \sum_n \varphi_1(d_1)\varphi_2(d_2)\varphi_3(d_3),$$

the last summation extending to all triads (d_1, d_2, d_3) such that $n = d_1d_2d_3$. With this notation, and (p, q, r), (i, j, k) any permutations of $(1, 2, 3)$, the following is obvious:

$$\sum_n [\varphi_p(d) \sum_\delta \varphi_q(\delta_1)\varphi_r(\delta_2)] = \sum_n \varphi_i(d_1)\varphi_j(d_2)\varphi_k(d_3).$$

Clearly this may be extended to any number of functions $\varphi_1, \varphi_2, \cdots, \varphi_r$. It will be found that the case $r = 4$ is required in some of the verifications:

$$\sum_n [\sum_d \varphi_1(d_1)\varphi_2(d_2) \sum_\delta \varphi_3(\delta_1)\varphi_4(\delta_2)] = \sum_n \varphi_1(d_1)\varphi_2(d_2)\varphi_3(d_3)\varphi_4(d_4),$$

in the first of which $\sum_n, \sum_d, \sum_\delta$ refer to all (d, δ), (d_1, d_2), (δ_1, δ_2) respectively such that $n = d\delta$, $d = d_1d_2$, $\delta = \delta_1\delta_2$; and $\sum_n$ in the second to all tetrads (d_1, d_2, d_3, d_4) such that $n = d_1d_2d_3d_4$. Also it is evident that

$$\sum_n [\sum_d \varphi_p(d_1)\varphi_q(d_2) \sum_\delta \varphi_r(\delta_1)\varphi_s(\delta_2)] = \sum_n [\varphi_i(d) \sum_\delta \varphi_j(\delta_1)\varphi_k(\delta_2)\varphi_l(\delta_3)],$$

where in the second, $\sum_n, \sum_\delta$ refer respectively to all (d, δ) such that $n = d\delta,$

and to all $(\delta_1, \delta_2, \delta_3)$ such that $\delta = \delta_1\delta_2\delta_3$; and (p, q, r, s), (i, j, k, l) are any permutations of $(1, 2, 3, 4)$. By the method of § 3 it is easily shown that if $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are factorable, then each of the multiple sums in this paragraph is a factorable function of n.

5. It is known that if $g_1(n)$, $g_2(n)$, $g_3(n)$ are factorable, then a factorable $f(n)$ exists such that

$$(6) \qquad \sum_n f(d)g_1(\delta) = \sum_n g_2(d)g_3(\delta).$$

Moreover, if $f(n)$, $g(n)$ both satisfy (6), then $f(n) = g(n)$ for all n; viz., (6) has a unique factorable solution. The following special case is of importance presently. Write $u_r(n) \equiv n^r$; then clearly $u_r(n)$ is a factorable function of n, and (5) may be written in the form

$$\sum_n \mu(d)u_0(\delta) = 0, \qquad n > 1.$$

Hence, if $g(n)$ is given and factorable, a unique factorable $f(n)$ may always be found such that

$$\sum_1 f(d)g(\delta) = 1; \qquad \sum_n f(d)g(\delta) = 0,\ n > 1.$$

It suffices to observe that $f(n)$ is uniquely determined by

$$\sum_n f(d)g(\delta) = \sum_n \mu(d)u_0(\delta).$$

6. Returning to § 1, let $\epsilon(n) = 1$ or 0 according as n is or is not a square. Clearly $\epsilon(n)$ is factorable. Replace (2) by its equivalent,

$$(7) \qquad T(n) = \sum_n \epsilon(d)P(\delta);$$

and hence, for δ any divisor of n,

$$(8) \qquad T(\delta) = \sum_\delta \epsilon(\delta_1)P(\delta_2).$$

Multiply (8) throughout by $f(d)$, where d is the conjugate of δ, and sum with respect to all pairs (d, δ). By § 4 the result may be written

$$(9) \qquad \sum_n T(d)f(\delta) = \sum_n [P(d) \sum_\delta \epsilon(\delta_1)f(\delta_2)].$$

Hence if $f(n)$ is determined as the solution of

$$(10) \qquad \sum_n f(d)\epsilon(\delta) = \sum_n u_0(d)\mu(\delta),$$

we shall have by the last of § 5 the following unique expression for $P(n)$ in terms of $T(n)$:

$$(11) \qquad P(n) = \sum_n T(d)f(\delta).$$

Write $\mu(n) \times \mu(n) \equiv \mu^2(n)$; then $\mu(n)$ being factorable, so also by the last of § 4 is $\sum_n \mu(d)\mu^2(\delta)$, since $\mu^2(n)$ obviously is factorable; and as in § 3 it may be verified without difficulty that the factorable function of n,

$$\sum_u [\epsilon(d) \sum_\delta \mu(\delta_1)\mu^2(\delta_2)] = 1 \text{ or } 0$$

according as $n = 1$ or $n > 1$; that is,

$$f(n) = \sum_n \mu(d)\mu^2(\delta) \tag{12}$$

is the required solution of (10). Finally, then the inversion (11) of (2) may be written

$$P(n) = \sum_n [T(d) \sum_\delta \mu(\delta_1)\mu^2(\delta_2)]. \tag{13}$$

7. To reduce (11), (13) to forms whose arithmetical significance is immediate, consider $f(n)$, which henceforth shall denote the function defined in (12). As in § 3, on verifying the statement for p^a, it is clear that $f(n)$ vanishes unless n is the square of a simple number, when the value is $+1$ or -1 according as the multiplicity of $\sqrt{n}$, or what is the same thing, the multiplicity of n, is even or odd. Hence (11), (13) may be paraphrased: The number of proper representations of n in (1) is equal to the sum of the total numbers of representations in (1) of all those divisors of n whose conjugates are squares of simple numbers of even multiplicity, diminished by the sum of the total numbers of representations in (1) of all those divisors of n whose conjugates are squares of simple numbers of odd multiplicity. It follows that the second sum never exceeds the first.

8. By (12) or § 7, for p prime we have

$$f(p) = 0; \qquad f(p^2) = -1; \qquad f(p^a) = 0,\ a > 2. \tag{14}$$

Combined with $f(1) = 1$, (14) has several important consequences which we proceed to develop. Let $\gamma(x)$ denote any function of x which vanishes when x is not a positive integer; and note that until further restricted, $\gamma(x)$ is not necessarily factorable. Call $\Gamma(n)$ defined by

$$\Gamma(n) = \sum_n \gamma(d)f(\delta) \tag{15}$$

the conjugate of $\gamma(n)$. *We shall always denote the conjugate of a given function by capitalizing*; thus the conjugates of $g(n)$, $\zeta_r(n)$, $\bar{\xi}_r'(n)$, $\cdots$ are $G(n)$, $Z_r(n)$, $\bar{\Xi}_r'(n)$, $\cdots$ respectively. Consider $\Gamma(p^a)$, where p is prime. By (15),

$$\Gamma(p^a) = \gamma(1)f(p^a) + \gamma(p)f(p^{a-1}) + \cdots + \gamma(p^{a-1})f(p) + \gamma(p^a)f(1); \tag{16}$$

and hence from (14),

$$\Gamma(p) = \gamma(p); \qquad \Gamma(p^a) = \gamma(p^a) - \gamma(p^{a-2}),\ a > 1; \tag{17}$$

both of which are included in the second, without the condition $a > 1$, since $\gamma(p^{-1}) = 0$. Again, if n is prime to p,

$$\begin{aligned}\Gamma(p^a n) &= \sum_n [\sum_{s=0}^{a} \gamma(p^s d) f(p^{a-s}\delta)] \\ &= \sum_n [\{\gamma(d)f(p^a) + \gamma(pd)f(p^{a-1}) + \cdots + \gamma(p^{a-1}d)f(p) \\ &\qquad + \gamma(p^a d)f(1)\}f(\delta)],\end{aligned}$$

the last on noticing that p^β and δ being relatively prime and f factorable, $f(p^\beta \delta) = f(p^\beta)f(\delta)$. Whence, on applying (14),

$$(18) \qquad \Gamma(p^a n) = \sum_n [\{\gamma(p^a d) - \gamma(p^{a-2}d)\}f(\delta)],$$

for p prime and not a divisor of n. By repeated application of (18) we get a remarkable symbolic form of the inversion (11). Let

$$n = p^a q^b \cdots r^c \equiv \Pi p^a$$

be the resolution of n into its prime factors; and let

$$\Pi[\gamma(p^a) - \gamma(p^{a-2})], \qquad \Pi'[\gamma(p^a) - \gamma(p^{a-2})]$$

denote respectively the ordinary product

$$[\gamma(p^a) - \gamma(p^{a-2})][\gamma(q^b) - \gamma(q^{b-2})] \cdots [\gamma(r') - \gamma(r^{c-2})],$$

and the like taken symbolically as follows: After distribution, each term in Π', such for example as $\gamma(p^a)\gamma(q^b) \cdots \gamma(r^{c-2})$, is to be replaced by the γ of the product of the several arguments of the γ's in that term, e.g., the particular term selected is to be replaced by $\gamma(p^a q^b \cdots r^{c-2})$; and the signs are to be as determined by the formal multiplication. Then from (18) we infer, on putting $n = q^b n_1$ where n_1 is prime to q, reapplying (18) and continuing thus until all the distinct prime powers $q^b, \cdots, r^c$ are exhausted,

$$(19) \qquad n = \Pi p^a, \qquad \Gamma(n) = \Pi'[\gamma(p^a) - \gamma(p^{a-2})].$$

The complete induction is immediate from (18), and need not be written out. Hence* from (11), (15), (19),

$$(20) \qquad n = \Pi p^a, \qquad P(n) = \Pi'[T(p^a) - T(p^{a-2})].$$

9. The special cases of (19), (20) in which $\gamma(n)$, $T(n)$ are factorable have a particular interest and importance. The $T(n)$ may be divided into two classes according as they are or are not factorable. The first includes all of the classical theorems of Gauss, Jacobi, Eisenstein, H. J. S. Smith and Liouville concerning representations of numbers as sums of

* Either (20) or its equivalent in § 7 may be proved from the definitions of $T(n)$, $P(n)$ by what H. J. S. Smith (Papers, I, p. 36) called the principle of cross-classification. The same principle gives also (5) and (19).

2, 4, 6, 8, 10 or 12 squares, with the exception of two additional theorems stated by Liouville, and proved by Glaisher, using elliptic functions, concerning 10 and 12 squares; it includes also the theory of those quadratic forms other than sums of squares to which the processes of elliptic functions are naturally adapted. Also when $T(n)$ is factorable, $T(n)$, $P(n)$ may both be calculated in finite form directly from the real divisors of n alone, without the invention of other functions, always more or less complicated, depending upon the representation of numbers in the given quadratic form in one of lower order. For forms other than sums of squares, the first three factorable $P(n)$, $T(n)$ theorems were stated by Eisenstein in his famous memoir, *Neue Theoreme der höheren Arithmetik* (*Crelle*, 35 (1847), p. 134); but the great mass of known results in this direction is due to Liouville, cf. § 12, footnote. It may be shown* that Liouville's 'formules générales' are equivalent to elementary identities in elliptic functions, whence they follow by a simple method of paraphrase. Hence all of Liouville's $P(n)$, $T(n)$ theorems ultimately depend upon the elements of elliptic functions, Pepin† having deduced the $T(n)$ results from the formules générales, and the $P(n)$ being consequences of these, as we shall presently indicate. The relation of Eisenstein's results to elliptic functions will be glanced at in § 14, footnote. It seems, in short, as lately suggested by Mordell,‡ that elliptic functions may have played a greater part in the discovery of many theorems than has been commonly supposed. In speaking of his $P(n)$ results, Liouville remarks (*J. des Math.* (2), 7 (1862), p. 16), "Il y a du reste à ce sujet, une méthod générale qui s'offre d'elle-même." Since the solution (12) is unique, (11) or its equivalent (13) must be what Liouville had in mind. We shall now examine the factorable case in some detail; (11), (20) apply to any case, factorable or not.

10. For $G(n)$ the conjugate of factorable $g(n)$, and for $T(n)$ factorable, we have from (19), (20),

$$(21) \qquad n = \Pi p^a, \qquad G(n) = \Pi[g(p^a) - g(p^{a-2})];$$

$$(22) \qquad n = \Pi p^a, \qquad P(n) = \Pi[T(p^a) - T(p^{a-2})].$$

For many forms (1) it has been found necessary or convenient to distinguish several $T(n)$ according to special factors of n; thus§ for m prime to 3, and

$$n = 2^\alpha 3^\beta m = x^2 + y^2 + z^2 + 3u^2$$

* In a series of papers presented to the American Math. Society, Oct., 1918, Liouville's general formulæ are derived incidentally.

† Journal des Math., 1890, pp. 1–64.

‡ Quarterly Journal Math., 48 (1917–1918), No. 189.

§ For the details cf. Liouville, J. des Math. (2), 8 (1863), pp. 105–114; 193–204.

the $T(n)$ take different forms (which, however, may all be included in one general formula), according as $\alpha = 0$, $\alpha = 1$, $\alpha > 1$, and $\beta = 0$, $\beta = 1$, $\beta > 1$. We say that $T(n)$ for this form has special characters with respect to the primes 2, 3. Let $p_1, p_2, \cdots, p_r$ be the primes with respect to which, for a given form, $T(n)$ has special characters. Then $P(n)$ is most readily investigated either by (18) or by

$$n = n_2 \prod_1^r p_i^{a_i}, \qquad P(n) = P(n_2) \prod_1^r [T(p_i^{a_i}) - T(p_i^{a_i-2})],$$

wherein n_2 is prime to p_i ($i = 1, \cdots, r$). Examples will be found in the illustrations. The importance of (21) is that for $T(n)$ factorable, $P(n)$ is the conjugate of a factorable function; hence we require such conjugates for (1).

11. A few $g(n)$ occur repeatedly in determinations of $T(n)$ for (1). All those in the literature are included in the following, or in simple modifications of them which it is unnecessary to consider here. Hence we shall find their corresponding $G(n)$, the notation being that of § 10, in order that the necessary data for writing down the corresponding $P(n)$ by (21), (22) may be readily accessible. By the usual convention the value of $(a \mid b)$, the Legendre-Jacobi symbol, is zero if a, b are not relatively prime, and $(a \mid b)$ is non-existent when b is even. In the following list the Π-notation has the same significance as in § 8; the Π-forms of the $g(n)$ are immediately evident from the definitions of the specific $g(n)$; and the deduction from these of the corresponding conjugates will be sufficiently clear from the full derivation of one of them. We recall that m is positive and odd. That all of the functions except λ are factorable is clear from their definitions.

(i) Let l denote an odd positive or negative constant integer prime to m, and define the ω, ω' functions by

$$\omega_r(m, l) = \sum_m (d \mid l) d^r, \qquad \bar{\omega}_r(l, m) = \sum_m (l \mid d) d^r,$$
$$\omega_r'(m, l) = \sum_m (\delta \mid l) d^r, \qquad \bar{\omega}_r'(l, m) = \sum_m (l \mid \delta) d^r.$$

Hence for $m = \Pi p^a$, we have:

$$\omega_r'(m, l) = \Pi \left[\frac{p^{r(a+1)} - (p^{a+1} \mid l)}{p - (p \mid l)} \right]; \qquad \bar{\omega}_r'(l, m) = \Pi \left[\frac{p^{r(a+1)} - (l \mid p^{a+1})}{p^r - (l \mid p)} \right];$$

and it is easily seen that

$$\omega_r(m, l) = (m \mid l) \omega_r'(m, l); \qquad \bar{\omega}_r(l, m) = (l \mid m) \bar{\omega}_r'(l, m).$$

Observing that $(p^{a+1} \mid l) = (p^{a-1} \mid l)$, and hence

$$\omega_r'(p^a, l) - \omega_r'(p^{a-2}, l) = \frac{p^{r(a+1)} - p^{r(a-1)}}{p^r - (p \mid l).} \equiv p^{ra}\left[1 + (p \mid l)\frac{1}{p^r}\right],$$

we have from (21) for the conjugate of $\omega_r'(m, l)$,

$$\Omega_r'(m, l) \equiv \Pi[\omega_r'(p^a, l) - \omega_r'(p^{a-2}, l)] = m^r\Pi\left[1 + (p|l)\frac{1}{p^r}\right].$$

Any conjugate may be found in the same way. We get thus

$$\bar{\Omega}_r'(l, m) = m^r\Pi\left[1 + (l|p)\frac{1}{p^r}\right] \equiv m^r\Pi\left[1 + (-1)^{\frac{1}{4}(l-1)(p-1)}(p|l)\frac{1}{p^r}\right],$$

the last on using the extended law of quadratic reciprocity; and

$$\Omega_r(m, l) = (m|l)\Omega_r'(m, l); \qquad \bar{\Omega}_r(l, m) = (l|m)\bar{\Omega}_r'(l, m).$$

The first form of $\bar{\Omega}_r'(l, m)$ presents itself directly in the consideration of (1); the second is better adapted to computation, and is equivalent to that occurring (for special values of r, l) in the writings of Eisenstein, Liouville and others. With these are two companions for the even case:

$$\text{(ii)} \qquad \omega_r(m) = \sum_m (2|d)d^r, \qquad \omega_r'(m) = \sum_m (2|\delta)d^r.$$

From these definitions we find as above,

$$\Omega_r'(m) = (2|m)\Omega_r(m) = m^r\Pi\left[1 + (2|p)\frac{1}{p^r}\right].$$

From the definitions it is clear that $\omega_r(m)$ is the sum of the rth powers of all those divisors of m which are of either form $8k \pm 1$, diminished by the like sum for the divisors of either form $8k \pm 3$; $\omega_r'(m)$ is the sum of the rth powers of all those divisors whose conjugates are of either form $8k \pm 1$, diminished by the like sum for the divisors whose conjugates are of either form $8k \pm 3$. Similarly, *for l prime*, $\omega_r(m, l)$ is the sum of the rth powers of all those divisors of m that are quadratic residues of l, diminished by the like sum for the divisors that are quadratic non-residues; $\bar{\omega}_r(l, m)$ is the sum of the rth powers of all those divisors of m of which l is a quadratic residue, diminished by the like sum for the divisors of which l is a quadratic non-residue; and $\omega_r'(m, l)$, $\bar{\omega}_r'(l, m)$ are the corresponding functions in which the divisors are segregated into classes according to the quadratic characters of their conjugates.

(iii) For $l = -1$ the ω-functions take important forms which, as they occur so frequently, are denoted by special letters. These appear first in the cases when (1) degenerates to a sum of squares; thus, they are familiar through the investigations of Jacobi, Eisenstein, H. J. S. Smith and Glaisher for 2, 6, 10, 14 and 18 squares; they also enter when (1) is a sum of 3, 7 or 11 squares.*

* In a paper presented to the American Mathematical Society, April, 1919, the $T(n)$ are given in the form of finite sums of the functions defined in § 11 when (1) is a sum of 3, 5, 7, 9, 11 or 13 squares, the arguments of the functions forming recurring series of the second order, and the number of odd squares in the representations being either pre-assigned or arbitrary.

Write

$$\bar{\omega}_r(-1, m) \equiv \xi_r(m), \qquad \bar{\omega}_r'(-1, m) \equiv \xi_r'(m);$$

whence

$$\xi_r'(m) = (-1)^{(m-1)/2}\xi_r(m), \qquad \Xi_r(m) = (-1)^{(m-1)/2}\Xi_r'(m),$$

$$\Xi_r'(m) = m^r\Pi\left[1 + (-1|p)\frac{1}{p^r}\right] = m^r\Pi\left[1 + (-1)^{(p-1)/2}\frac{1}{p^r}\right];$$

and $\xi_r(m)$ is the excess of the sum of the rth powers of all those divisors of m that are of the form $4k + 1$ over the like sum for the divisors of the form $4k - 1$; $\xi_r'(m)$ is the similar function in which the conjugates of the divisors are of the respective forms $4k + 1, 4k - 1$. For $n = 2^\alpha m$ we have

$$\xi_r(n) = \xi_r(m), \qquad \xi_r'(n) = 2^{\alpha r}\xi_r'(m);$$

and noticing that by an obvious extension $\omega_r(n, l)$, $\omega_r'(n, l)$ may be defined for $\alpha > 0$, the following conjugates:

$$\Xi_r(2m) = \Xi_r(m); \qquad \Xi_r(2^\alpha m) = 0,\ \alpha > 1;$$

$$\Xi_r'(2m) = 2^r\Xi_r'(m); \qquad \Xi_r'(2^\alpha m) = 2^{(\alpha-2)r}(2^{2r} - 1)\Xi_r'(m),\ \alpha > 1.$$

(iv) For $n = 2^\alpha m$, $\alpha \geqq 0$, and $n = \Pi p^a$ the resolution of n into prime factors, $\zeta_r(n)$, $\zeta_r'(n)$ the respective sums of all, of the odd divisors of n, we have in the same way:

$$\zeta_r(n) = \Pi\left[\frac{p^{r(a+1)} - 1}{p^r - 1}\right]; \qquad \zeta_r'(n) = \zeta_r(m);$$

$$Z_r(n) = n^r\Pi\left[1 + \frac{1}{p^r}\right]; \qquad Z_r(2^\alpha m) = 2^{(\alpha-1)r}(2^r + 1)Z_r(m),\ \alpha > 0;$$

$$Z_r'(2m) = Z_r'(m) = Z_r(m); \qquad Z_r'(2^\alpha m) = 0,\ \alpha > 1.$$

(v) Closely related to these are the two following: $\alpha_r(n)$, = the sum of the rth powers of all those divisors of n whose conjugates are odd; and the non-factorable $\lambda_r(n)$ defined by

$$\lambda_r(n) = [2(-1)^n + 1]\zeta_r'(n).$$

For the respective conjugates we find, using (18) for Λ_r:

$$A_r(m) = Z_r(m); \qquad A_r(2m) = 2^rZ_r(m);$$

$$A_r(2^\alpha m) = 2^{r(\alpha-2)}(2^{2r} - 1)Z_r(m),\ \alpha > 1;$$

$$\Lambda_r(m) = -Z_r(m); \qquad \Lambda_r(2m) = 3\,Z_r(m); \qquad \Lambda_r(4\,m) = 4Z_r(m);$$

$$\Lambda_r(2^\alpha m) = 0,\ \alpha > 2.$$

12. As a first illustration of the formulæ we take Eisenstein's theorems.*

* Eisenstein, Crelle, 35 (1847), pp. 134–135.

We shall assume the $T(m)$, all of which, occurring among the forms considered by Liouville,* are proved in Pepin's memoir, and from them deduce the $P(m)$. The forms are

$$x^2 + y^2 + z^2 + 3u^2; \quad x^2 + y^2 + 2z^2 + 2uz + 2u^2; \quad x^2 + y^2 + z^2 + 5u^2;$$

and when m is prime to 3 for the first two, prime to 5 for the third, the several cases of Eisenstein's $T(m)$ may be written for the three forms respectively:

$$T(m) = A\bar{\omega}_1(3, m), B\bar{\omega}_1(3, m), C\omega_1(m, 5);$$

$$A = [2(-1|m) + 1][3(-3|m) - 1],$$

$$B = [2(-1|m) - 1][3(-3|m) + 1],$$

$$C = 5(m|5) + 1.$$

It is convenient to separate A into two cases, A' for $m \equiv 1, 7 \bmod 12$, A'' for $m \equiv 5, 11 \bmod 12$:

$$A' = 2[1 + 2(3|m)], \qquad A'' = 4[2(3|m) - 1].$$

Hence for the first form when $m \equiv 1, 7 \bmod 12$, we have by (22) and § 11 (i),

$$P(m) = 2[\bar{\Omega}_1(3, m) + 2\bar{\Omega}_1'(3, m)] = 2[(3|m) + 2]\bar{\Omega}_1'(3, m);$$

and when $m \equiv 5, 11 \bmod 12$,

$$P(m) = 4[2\bar{\Omega}_1'(3, m) - \bar{\Omega}_1(3, m)] = 4[2 - (3|m)]\bar{\Omega}_1'(3, m).$$

It is readily seen that both are included in either of the single formulæ, the second following at once from the first by § 11 (i),

$$P(m) = (3|m)A\bar{\Omega}_1'(3, m), \qquad P(m) = A\bar{\Omega}_1(3, m),$$

either agreeing with the four cases stated by Eisenstein. Similarly, on separating B in the same way mod 12 into

$$B' = 4[-1 + 2(3|m)], \qquad B'' = 2[1 + 2(3|m)],$$

we find for the second form

$$P(m) = (3|m)B\bar{\Omega}_1'(3, m), \qquad P(m) = B\bar{\Omega}_1(3, m);$$

and for the third, without separation,

$$P(m) = (m|5)C\Omega_1'(m, 5), \qquad P(m) = C\Omega_1(m, 5).$$

Thus in all cases we may pass from T to P by changing ω into Ω, a special case of a general theorem which need not concern us here.

* Journal des Math. (2), vols. 4–11 (1859–1866).

13. The treatment of special characters (§ 10) is exemplified by Liouville's*

$$x^2 + 2y^2 + 4z^2 + 4u^2,$$

for which he states

$$T(m) = 2\omega_1'(m); \qquad T(2^\alpha m) = 2[2^\alpha - (2\,|\,m)]\omega_1'(m),\ \alpha > 0.$$

Hence by § 11 (ii), $P(m) = 2\Omega_1'(m)$; and for $\alpha > 0$, as in deriving (18):

$$P(2^\alpha m) = \sum_m [T(d)f(2^\alpha\delta) + T(2d)f(2^{\alpha-1}\delta) + \cdots + T(2^\alpha d)f(\delta)];$$

whence for $\alpha > 1$,

$$P(2m) = \sum_m T(2d)f(\delta), \qquad P(2^\alpha m) = \sum_m [\{-T(2^{\alpha-2}d) + T(2^\alpha d)\}f(\delta)];$$

and as in § 12 on substituting the values of $T(2d)$, $T(2^\alpha d)$ for $\alpha = 2, 3, 4, \cdots$ we find

$$P(2m) = 2[2 - (2\,|\,m)]\Omega_1'(m); \qquad P(4m) = 2[3 - (2\,|\,m)]\Omega_1'(m);$$
$$P(2^\alpha m) = 3\cdot 2^{\alpha-1}\Omega_1'(m),\ \alpha > 2,$$

agreeing with the statements of Liouville, loc. cit., § 4.

14. As a last example, let us find the $P(n)$ when (1) is a sum of 4, 6, 8, 10 or 12 squares, from the known factorable $T(n)$, all of the latter having been most simply derived by elliptic functions.†

(i) For four squares we have

$$T(m) = 8\zeta_1(m); \qquad T(2^a m) = 24\zeta_1(m),\ a > 0.$$

* J. des Math. (2), 7 (1862), pp. 62–64. The $P(n)$ for the following papers in the same volume may be proved as in § 13: 145–147; 148–149; 201–204; 205–208.

† See the cited papers of Glaisher and Mordell for theorems and references, to which add the following for 10, 12 squares: Liouville, J. des Math. (2), 11 (1866), pp. 1–8; 6 (1861), pp. 369–377; 233–238. Proofs for the remarkable general $T(n)$ formulæ in these papers of Liouville will appear shortly in the Bulletin of the American Math. Society. Combined with those proofs, the formulæ for $Z_r(n)$, $\Xi_r(n)$, $\Xi_r'(n)$ in § 11 above give at once the proofs for Liouville's general $P(n)$ formulæ, ibid., pp. 373–376. The relation between the 10 and 12 square theorems and the rest of Liouville's $T(n)$ results is simple and striking: the "formules générales" whence Pepin proved the latter are direct paraphrases of trigonometric identities arising from elliptic identities such as $sn^2x = snx \times snx$, when for snx, sn^2x are substituted their Fourier developments and coefficients of q^n equated; the general theorems whence Liouville deduces his 10 and 12 square results come from precisely the same identities when for the elliptic functions and their powers are substituted their power-series expansions, and coefficients of x^n equated. Thus all these apparently diverse results are seen from the standpoint of elliptic functions to be ultimately the same, differing only in algebraic details. The use of the "formules générales" can be avoided entirely, by assigning x the values $\pi/2$, $\pi/3$, $\pi/4$, $\pi/5$, $\pi/6$, $\pi/8$ in the trigonometric identities, the procedure thence onward being obvious from the first sections of Pepin's memoir. As further indicating the connection with elliptic functions of Eisenstein's 10-square and other results, *all* of his assertions concerning 10 squares are proved by Glaisher's formula (i), Q. J. Math., 38 (1906–7), p. 22. The $P(n)$ formulæ for 4 and the $P(m)$ for 6 squares found above agree with those determined arithmetically, cf. Bachmann, Quadr. Formen, pp. 602, 652.

Hence by § 10 and § 11 (iv),

$$P(m) = 8Z_1(m); \qquad P(2m) = 24Z_1(m); \qquad P(4m) = 16Z_1(m);$$
$$P(2^a m) = 0,\ a > 2.$$

(ii) For six squares the known $T(n)$ is

$$T(n) = 4[4\xi_2'(n) - \xi_2(n)];$$

and therefore by § 11 (iii):

$$P(m) = 4[\ 4\Xi_2'(m) - \Xi_2(m)] = 4[\ 4 - (-1\,|\,m)]\Xi_2'(m);$$
$$P(2m) = 4[16\Xi_2'(m) - \Xi_2(m)] = 4[16 - (-1\,|\,m)]\Xi_2'(m);$$
$$P(2^a m) = 15\cdot 2^{2a}\Xi_2'(m),\ a > 1.$$

(iii) For eight squares, $T(n) = -16(-1)^n[2\zeta_3'(n) - \zeta_3(n)]$. Hence

$$T(m) = 16\zeta_3(m); \qquad 7T(2^a m) = 16[2^{3(a+1)} - 15]\zeta_3(m),\ a > 0;$$
$$P(m) = 16Z_3(m); \qquad P(2m) = 112Z_3(m);$$
$$P(2^a m) = 144.2^{3(a-1)}Z_3(m),\ a > 1.$$

(iv) For ten squares the only factorable case is for $n = 2^b m$, $b \geqq 0$, $m \equiv 3 \bmod 4$:

$$T(n) = 4[\xi_4(n) + 16\xi_4'(n)].$$

To find $P(n)$ we have, by the general formulæ,

$$P(m) = \sum_m T(d)f(\delta); \qquad P(2m) = \sum_m T(2d)f(\delta);$$
$$P(2^a m) = \sum_m [T(2^a d) - T(2^{a-2}d)]f(\delta),\ a > 1.$$

The divisors d remaining in these formulæ after reduction will all be of the prescribed form $2^b(4k + 3)$ when and only when m is a prime $\equiv 3$ mod 4. Hence *in this case only* we get the following for the numbers of proper representations as a sum of ten squares:

$$P(m) = T(1)f(m) + T(m)f(1) = T(m),$$

the only divisors of m being $1, m$. For this value of m, $\xi_4'(m) = -1 + m^4$, $\xi_4(m) = 1 - m^4$; hence $P(m) = 60(m^4 - 1)$. Similarly

$$P(2m) = T(2)f(m) + T(2m)f(1) = T(2m);$$
$$P(2^a m) = T(2^a m) - T(2^{a-2}m),\ a > 1;$$

and hence, after obvious reductions,

$$P(2m) = 1020(m^4 - 1); \qquad P(2^a m) = 255.2^{2(2a-1)}(m^4 - 1),\ a > 1.$$

There are no similar theorems for m not a prime of the form $4k + 3$.

(v) For twelve squares a factorable $T(n)$ exists only when n is even:

$$31T(2^a m) = 24[21 + 2^{5a+1}\cdot 5]\zeta_5(m),\ a > 0.$$

Hence, at once by § 11 (v), we find $P(2m) = 264Z_5(m)$; a result stated without proof by Liouville in a letter to M. Besge (J. des Math. (2), 5 (1860), p. 145). For $a > 1$:

$$P(2^a m) = \sum_m [T(2^a d) - T(2^{a-2}d)]f(\delta).$$

Now $2^a d$, $2^{a-2}d$ will both be even when and only when $a > 2$; and we find, on simplifying,

$$P(2^{a+1}m) = 495\cdot 2^{5a-6}Z_2(m),\ a > 2.$$

15. The extensions of the fundamental formula (11) and § 7 to (3) are obvious and at present of slight interest: it is sufficient to replace $f(n)$ by the factorable $f_s(n)$, which vanishes unless n is the sth power of a simple number, in which case its value is ± 1 according as the multiplicity of n is even or odd.

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